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Powers of totally positive subgroups of fields

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ABSTRACT

For a formally real field F, a totally positive co-torsion subgroup S of $F^\times$ and $m \geqslant 1$, we prove that $(\sum S)^m \subseteq \sum S^m$. This extends a result of Becker. We prove several refinements of this fact related to higher Pythagoras numbers, real holomorphy rings, and sums with mixed powers.

1. Introduction

Let F be a field and let $F^\times$ be its multiplicative group. We assume that F is formally real, i.e., that -1 is not a sum of squares in F. In the papers [Bec78,Bec79a,Bec79b,Bec82] Becker discovered a list of remarkable facts concerning sums of nth powers in F, where n is an even positive integer. Among other things, he proved that for $S = (F^\times)^n$ and for every positive integer m one has

$$\left(\sum S\right)^m \leqslant \sum S^m. \tag{1.1}$$

Here $\sum A$ denotes the set of all non-zero sums from the subset A of F. This was later extended by Berr [Ber92] to sums of powers with mixed even exponents.

In particular, for $F = \mathbb{Q}(X_1, \ldots, X_k)$ one deduces from (1.1) the existence of identities

$$\left(\sum_{i=1}^{k} X_i^n\right)^m = \sum_{j=1}^{r} f_j(X_1, \ldots, X_k)^{nm} \tag{1.2}$$

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for some rational functions $f_j(X_1,\dots,X_k)\in F$. In fact, for $n=2$ the existence of such identities was proved non-constructively by Hilbert [Hil09] as a main step in his proof of the Waring conjecture (see [Ell71]). More constructive proofs were later given by Hausdorff [Hau09] and Stridsberg [Str12]. Still, explicit examples of identities of the form (1.2) are currently known only for some special values of n and m (see, e.g., [Rez92, p. 103]). For introductions to Becker's theory we refer to [Bec84], [Lam80, §12], and [Pow96].

In this paper we extend these results of Becker and Berr (in particular (1.1)) by showing that they hold more generally for subgroups S of $F^\times$, where

(1) S is **co-torsion** in $F^\times$ (i.e., $F^\times/S$ is a torsion group); and
(2) S is **totally positive**, i.e., $S\subseteq\sum(F^\times)^2$.

Recall that by a celebrated theorem of Artin, $\sum(F^\times)^2$ is the set of all totally positive elements of F, i.e., the elements which are positive with respect to all orderings on F. Of course, when n is even, the subgroup $(F^\times)^n$ satisfies (1) and (2).

In Section 7 we extend to this context other parts of Becker's theory, related to higher level Pythagorean fields, Pythagoras numbers, and their connections with the real holomorphy ring. Unlike the original approach to this theory, our proofs in the general case do not use orderings of higher level.

2. Formally real groups

Let F be a field and S a subgroup of $F^\times$. We say that S is a **formally real** subgroup if $-1\notin\sum S$. Thus $(F^\times)^2$ is a formally real subgroup of $F^\times$ if and only if F is a formally real field. The proof of the next lemma is straightforward (recall that, by definition, $0\notin\sum S$).

Lemma 2.1. *The following conditions are equivalent*:

(a) *S is formally real*;
(b) 0 *is not a (non-empty) sum of finitely many elements of S*;
(c) *$\sum S$ is additively closed.*

For S co-torsion, every multiplicatively closed set $S\subseteq S'\subseteq F^\times$ is a subgroup of $F^\times$; indeed, for every $a\in S'$ there is a positive integer n with $a^n\in S$, whence $a^{-1}=a^{n-1}(a^n)^{-1}\in S'$. Therefore, in this case (c) above means that $\sum S$ is an additively closed *subgroup* of $F^\times$. We also note that an additively closed subgroup of $F^\times$ cannot contain -1.

Next we rephrase [Bec79b, Cor. 3.5] as follows.

Theorem 2.2. *The following conditions on the field F are equivalent*:

(a) *F is formally real*;
(b) *there is an additively closed co-torsion subgroup of $F^\times$*;
(c) *there is a formally real co-torsion subgroup of $F^\times$.*

Proof. A subgroup S of $F^\times$ is additively closed and co-torsion if and only if $S\cup\{0\}$ is a *torsion preordering* in the sense of [Bec79b, §3]. Thus the equivalence of (a) and (b) is just [Bec79b, Cor. 3.5].

The equivalence of (b) and (c) follows from Lemma 2.1, in view of the preceding remark. □

Remark 2.3. For an example of a field F and a co-torsion formally real subgroup S of $F^\times$ with $F^\times/S$ of infinite exponent see [Bec79b, (3.6)].

3. Real valuations

Let v be a (Krull) valuation on the field F (for notions in valuation theory we refer to [Efr06]). We denote the valuation ring, valuation ideal, group of units, and residue field of v by O_v, $\mathfrak{m}_v$, $O_v^\times$, and $\bar{F}_v$, respectively. Given a subgroup S of $F^\times$, the set $\bar{S}_v = (\overline{S \cap O_v^\times})_v$ of all residues of $S \cap O_v^\times$ with respect to v is a subgroup of $\bar{F}_v^\times$. We call it the **push-down** of S to $\bar{F}_v$. When $\bar{S}_v$ is formally real, the ultrametric inequality becomes an equality on S, as the next lemma shows.

Lemma 3.1. *Let S be a subgroup of $F^\times$ and let v be a valuation on F with $\bar{S}_v$ formally real. For every $a_1, \ldots, a_n \in S$ we have*

$$v\left(\sum_{i=1}^n a_i\right) = \min_{1 \leqslant i \leqslant n} v(a_i).$$

Proof. We may assume without loss of generality that $v(a_1) \leqslant v(a_i)$ for all i. Then $b = 1 + \sum_{i=2}^n a_i a_1^{-1}$ lies in O_v. Furthermore, $\overline{a_i a_0^{-1}} \in \bar{S}_v \cup \{0\}$ for every i. Since $\bar{S}_v$ is formally real, $\bar{b} \neq \bar{0}$ in $\bar{F}_v$. Therefore $v(b) = 0$, as required. $\square$

One says that a valuation is **real** if its residue field is formally real.

Lemma 3.2. *Let v be a real valuation on F and let $a_1, \ldots, a_n \in F^\times$.*

(a) *$v(\sum_{i=1}^n a_i^2) \geqslant 0$ if and only if $v(a_i) \geqslant 0$ for all $1 \leqslant i \leqslant n$.*
(b) *For $a_1, \ldots, a_n$ totally positive, $v(\sum_{i=0}^n a_i) = \min_{0 \leqslant i \leqslant n} v(a_i)$.*

Proof. (a) The "if" part is clear. For the "only if" part, apply Lemma 3.1 with $S = (F^\times)^2$, noting that $\bar{S}_v = (\bar{F}_v^\times)^2$.

(b) By (a), $O_v \cap \sum (F^\times)^2$ consists of the non-zero sums of squares of elements of O_v. Consequently, the push-down of $\sum (F^\times)^2$ to $\bar{F}_v$ is $\sum (\bar{F}_v^\times)^2$. Now apply Lemma 3.1 with $S = \sum (F^\times)^2$. $\square$

Assume now that F is formally real. Then the trivial valuation is real. The intersection H_F of all real valuation rings on F is called the **real holomorphy ring**. As usual, we denote its set of invertible elements by $H_F^\times$. Following [Bec82], we denote

$$\mathbb{E}^+ = H_F^\times \cap \sum \left(F^\times\right)^2.$$

Note that it is a subgroup of $H_F^\times$.

Corollary 3.3. *For totally positive $a_1, \ldots, a_n \in F$ and for a positive integer m one has*

$$\frac{\sum_{i=1}^n a_i^m}{(\sum_{i=1}^n a_i)^m} \in \mathbb{E}^+.$$

Proof. We first note that since F is formally real and the a_i are totally positive, both the numerator and the denominator here are non-zero. Now for every real valuation v on F Lemma 3.2(b) gives

$$v\left(\sum_{i=1}^n a_i^m\right) = m \min_{1 \leqslant i \leqslant n} v(a_i) = v\left(\left(\sum_{i=1}^n a_i\right)^m\right).$$

The assertion follows. $\square$

4. The ring $A(S)$

Consider an additively closed subgroup S of $F^\times$. Then char $F=0$ and S contains all positive rational numbers. We define

$$A(S)=\bigcup_{n=1}^{\infty}\big((n-S)\cap(S-n)\big).$$

The set $A(S)$ contains $0,1,-1$ and is additively closed. The identity

$$nm+ab=\frac{1}{2}\big[(n+a)(m+b)+(n-a)(m-b)\big]$$

implies that $A(S)$ is also multiplicatively closed, hence is a ring.

We recall from [Bec79b] the following main properties of $A(S)$.

(i) For $s\in S$ the identity

$$\frac{1}{1+s}+\frac{1}{1+s^{-1}}=1$$

implies that both summands are in $S\cap(1-S)$, whence are in $A(S)$. Conversely, for every $a\in A(S)$ we can choose a positive integer n with $n\pm a\in S$. Then

$$s:=\frac{n-a}{n+a}\in S,\qquad a=\frac{2n}{1+s}-n.$$

Therefore the ring $A(S)$ is generated over $\mathbb{Z}$ by all elements of the form $1/(1+s)$ with $s\in S$.

(ii) Assume in addition that S is a co-torsion subgroup of $F^\times$ and let K be the subfield it generates. Then $F^\times/K^\times$ is a torsion group. But by [Bec79b, Lemma 3.1] this can happen for a field extension of characteristic 0 only when $F=K$. Now for $s\in S$ one has $s=(1+s)/(1+s^{-1})$. In view of (i), this shows that the fractions field of $A(S)$ contains S, and therefore must equal F.

(iii) For S co-torsion, $A(S)$ is a Prüfer domain, i.e., its localizations at maximal ideals are valuation rings on the fraction field F [Bec79b, Th. 3.7(i)]. Since every domain is the intersection of its localizations at maximal ideals [Mat80, 1.H], $A(S)$ is an intersection of valuations rings on F.

(iv) If v is a valuation on F such that $A(S)\subseteq O_v$, then by (i), $v(1+s)\leqslant 0$ for all $s\in S$. Hence $\bar{S}_v$ is a formally real subgroup of $\bar{F}_v^\times$. Furthermore, when S is co-torsion in $F^\times$, the subgroup $\bar{S}_v$ is co-torsion in $\bar{F}_v^\times$. By Theorem 2.2, $\bar{F}_v$ is then formally real, i.e., v is a real valuation.

See [BBG99, §1] for some additional properties of $A(S)$.

Proposition 4.1. *Let F be a field and let S be a totally positive and additively closed co-torsion subgroup of $F^\times$. Then $A(S)=H_F$.*

Proof. In light of Theorem 2.2, F is formally real. By facts (iii) and (iv), $A(S)$ is an intersection of valuation rings on F corresponding to real valuations. Hence $A(S)\supseteq H_F$.

Conversely, for every real valuation v and for every $s\in S$ we have $v(1+s)\leqslant 0$, by Lemma 3.2(b). Hence $1/(1+s)\in H_F$. We conclude from (i) that $A(S)\subseteq H_F$. □

5. The representation theorem

Let A be a unitary ring and let $\mathrm{Hom}(A,\mathbb{R})$ be the set of all unitary ring homomorphisms $A\to\mathbb{R}$. For $a\in A$ let $\hat{a}:\mathrm{Hom}(A,\mathbb{R})\to\mathbb{R}$ be the evaluation map $\hat{a}(h)=h(a)$. We equip $\mathrm{Hom}(A,\mathbb{R})$ with the weakest topology making all maps $\hat{a}$ continuous.

Next let M be a subset of A such that

$$0, 1 \in M, \quad -1 \notin M, \quad M + M \subseteq M, \quad M \cdot M \subseteq M. \tag{5.1}$$

We assume that M is *Archimedean* in the sense that for every $a \in A$ there exists a positive integer n such that $n \cdot 1 - a \in M$. Let M_{div} be the divisible hull of M in A, i.e., the set of all $a \in A$ such that $na \in M$ for some positive integer n. Let X_M be the set of all $h \in \mathrm{Hom}(A, \mathbb{R})$ such that $h(M) \subseteq \mathbb{R}_{\geqslant 0}$. Note that it is closed in $\mathrm{Hom}(A, \mathbb{R})$. Also, for every $a \in A$ there exists $n_a \geqslant 1$ such that $n_a \cdot 1 \pm a \in M$. It follows that X_M embeds as a closed subset of $\prod_{a \in A} [-n_a, n_a]$ via the map $\prod_{a \in A} \hat{a}$. This makes X_M a compact topological space (see [Mar02, Prop. 1.1]).

As usual, let $C(X_M, \mathbb{R})$ be the ring of all continuous maps $X_M \to \mathbb{R}$. Let $C(X_M, \mathbb{R}_{\geqslant 0})$ be its subset consisting of all continuous maps into the non-negative reals. Finally, we define a ring homomorphism $\Phi : A \to C(X_M, \mathbb{R})$ by $\Phi(a) = \hat{a}|_{X_M}$. The following fact is a part of the Kadison–Dubois representation theorem [BS83,Mar02].

Theorem 5.1. *$\Phi^{-1}(C(X_M, \mathbb{R}_{\geqslant 0}))$ consists of all $a \in A$ such that for every non-negative integer n one has $1 + na \in M_{\mathrm{div}}$.*

Denoting the set of non-zero sums of squares in A by $\sum A^2$, we get:

Lemma 5.2. *In the above setup*:

(a) $A^\times \cap \Phi^{-1}(C(X_M, \mathbb{R}_{\geqslant 0})) \subseteq M_{\mathrm{div}}$.
(b) $A^\times \cap \sum A^2 \subseteq M_{\mathrm{div}}$.

Proof. (a) Let $a \in A^\times \cap \Phi^{-1}(C(X_M, \mathbb{R}_{\geqslant 0}))$. For every $h \in X_M$ we have $h(a) \geqslant 0$ and $h(a)h(a^{-1}) = h(1) = 1$. Hence $h(a) > 0$. The compactness of X_M yields a positive integer n with $h(a) > 1/n$ for all $h \in X_M$. It follows that

$$na \in 1 + \Phi^{-1}\big(C(X_M, \mathbb{R}_{\geqslant 0})\big) \subseteq M_{\mathrm{div}},$$

by Theorem 5.1. Thus $a \in M_{\mathrm{div}}$.

(b) One has $\sum A^2 \subseteq \Phi^{-1}(C(X_M, \mathbb{R}_{\geqslant 0}))$, so we apply (a). $\square$

Now let F be again a formally real field and let S be an additively closed subgroup of $F^\times$. Let $A = A(S)$ and let $M = (A \cap S) \cup \{0\}$. Then the assumptions (5.1) are satisfied. Further, M is Archimedean and $M = M_{\mathrm{div}}$. We obtain:

Corollary 5.3. *For an additively closed co-torsion subgroup S of $F^\times$ one has*

$$A(S)^\times \cap \sum \big(F^\times\big)^2 \subseteq S.$$

Proof. By facts (iii) and (iv) of Section 4, $A(S) = \bigcap_v O_v$ for some collection of real valuations v on F. By Lemma 3.2(a), for every real valuation v and for every $b_1, \ldots, b_n \in F^\times$, if $\sum_{i=1}^n b_i^2 \in O_v$, then $b_1, \ldots, b_n \in O_v$. Consequently, $A(S) \cap \sum (F^\times)^2 \subseteq \sum A(S)^2$. Now apply Lemma 5.2(b). $\square$

Remark 5.4. Corollary 5.3 actually contains the main implication (b) $\Rightarrow$ (a) in Becker's Theorem 2.2. Becker's original proof of this result is also based on the representation Theorem 5.1.

From Proposition 4.1 and Corollary 5.3 we deduce:

Corollary 5.5. *For F formally real, $\mathbb{E}^+$ is contained in every totally positive and additively closed co-torsion subgroup of $F^\times$.*

6. Totally positive sums and valuations

Let F be again a formally real field. We consider totally positive elements α_λ of $F^\times$ and totally positive co-torsion subgroups S_λ, $\lambda \in \Lambda$, of $F^\times$. Then $\sum S_\lambda$ is an additively closed and totally positive co-torsion subgroup of $F^\times$. Given $s \in S_\lambda$ one has $-1 \neq s/\alpha_\lambda \,(\in \sum (F^\times)^2)$, so we can choose $k = k(\lambda, s) \geqslant 1$ with $(1 + s/\alpha_\lambda)^k \in S_\lambda$. We set

$$\beta_\lambda(s) = \frac{s}{\alpha_\lambda(1 + s/\alpha_\lambda)^k}.$$

Note that $\beta_\lambda(s)$ is totally positive and $\beta_\lambda(s) S_\lambda = \alpha_\lambda^{-1} S_\lambda$. Moreover, for every real valuation v on F Lemma 3.2(b) gives

$$v\bigl(\beta_\lambda(s)\bigr) = v(s/\alpha_\lambda) - k \min\bigl\{0, v(s/\alpha_\lambda)\bigr\} \geqslant 0. \tag{6.1}$$

Hence $\beta_\lambda(s) \in H_F$.

The equivalence of (a) and (b) in the proposition below generalizes a part of [Bec79a, Satz 2.14] (for the single subgroup $(F^\times)^m$ with m even) and [Ber92, Th. 1.2] (for $S_\lambda = (F^\times)^{m_\lambda}$ with m_λ even). Our argument is inspired by Berr's proof of the latter result. Here $\sum_{\lambda \in \Lambda} \alpha_\lambda^{-1} \sum S_\lambda$ is the set of all non-zero sums of elements of $\bigcup_{\lambda \in \Lambda} \alpha_\lambda^{-1} S_\lambda$.

Proposition 6.1. *The following conditions are equivalent*:

(a) $1 \in \sum_{\lambda \in \Lambda} \alpha_\lambda^{-1} \sum S_\lambda$;
(b) *for every real valuation v on F there exists $\lambda \in \Lambda$ such that $v(\alpha_\lambda) \in v(S_\lambda)$*;
(c) *for every real valuation v on F there exist $\lambda \in \Lambda$ and $s \in S_\lambda$ such that $v(\beta_\lambda(s)) = 0$.*

Proof. (a) $\Rightarrow$ (b): Write $1 = \sum_{i=1}^r \alpha_{\lambda(i)}^{-1} \sum_{j=1}^{t(i)} s_{ij}$ with $\lambda(i) \in \Lambda$ and $s_{ij} \in S_{\lambda(i)}$. Since $\alpha_{\lambda(i)}, s_{ij}$ are totally positive, Lemma 3.2(b) gives i, j with $v(\alpha_{\lambda(i)}) = v(s_{ij}) \in v(S_{\lambda(i)})$.

(b) $\Rightarrow$ (c): Take $\lambda \in \Lambda$ and $s \in S_\lambda$ with $v(\alpha_\lambda) = v(s)$. By (6.1), $v(\beta_\lambda(s)) = 0$.

(c) $\Rightarrow$ (a): Let $\mathfrak{m}$ be a maximal ideal of $H_F = A(\sum (F^\times)^2)$ (see Proposition 4.1). By facts (iii) and (iv) of Section 4, $(H_F)_\mathfrak{m} = O_v$ for some real valuation v on F. By (c), $\mathfrak{m}_v$ cannot contain all the $\beta_\lambda(s)$, with $\lambda \in \Lambda$ and $s \in S_\lambda$. As $\mathfrak{m} \subseteq \mathfrak{m}_v$, neither can $\mathfrak{m}$. Since $\mathfrak{m}$ was arbitrary, there must exist $\lambda(1), \dots, \lambda(t) \in \Lambda$ and $s_i \in S_{\lambda(i)}$, $i = 1, \dots, t$, such that

$$1 \in \beta_{\lambda(1)}(s_1) H_F + \cdots + \beta_{\lambda(t)}(s_t) H_F.$$

Let

$$\gamma = \beta_{\lambda(1)}(s_1) + \cdots + \beta_{\lambda(t)}(s_t) \in H_F.$$

For every real valuation v on F and for $1 \leqslant i \leqslant t$, Lemma 3.2(b) gives $v(\gamma) \leqslant v(\beta_{\lambda(i)}(s_i))$. Hence $\beta_{\lambda(i)}(s_i)/\gamma \in H_F$, so $1 \in \gamma H_F$. It follows that $\gamma \in \mathbb{E}^+$. Now for each i we may apply Corollary 5.5 for the group $\sum S_{\lambda(i)}$ to obtain that $\mathbb{E}^+ \leqslant \sum S_{\lambda(i)}$. Consequently

$$1 = \sum_{i=1}^t \beta_{\lambda(i)}(s_i)\gamma^{-1} \in \sum_{i=1}^t \beta_{\lambda(i)}(s_i) \sum S_{\lambda(i)} = \sum_{i=1}^t \alpha_{\lambda(i)}^{-1} \sum S_{\lambda(i)},$$

proving (a). $\square$

For a single subgroup $S=S_\lambda$ we obtain in particular:

Theorem 6.2. *Let S be a totally positive co-torsion subgroup of $F^\times$. Then $\sum S$ consists of all totally positive $\alpha\in F^\times$ such that $v(\alpha)\in v(S)$ for every real valuation v on F.*

The next corollary extends [Bec79a, Satz 2.7, Kor. 2.8] and [BG95, Cor. 2.15] (which deal with subgroups $S=(F^\times)^n$).

Corollary 6.3. *The following conditions on a co-torsion subgroup S of $F^\times$ are equivalent:*

(a) $\sum(F^\times)^m=\sum S^m$ *for every even positive integer m;*
(b) $\sum(F^\times)^m=\sum S^m$ *for some even positive integer m;*
(c) *for every real valuation v one has $v(F^\times)=v(S)$.*

Proof. (a) $\Rightarrow$ (b): Trivial.

(b) $\Rightarrow$ (c): Let $\alpha\in F^\times$. By (b), $\alpha^m\in\sum S^m$. By Lemma 3.2(b), $v(\alpha^m)\in v(S^m)$. Since $v(F^\times)$ is torsion-free, $v(\alpha)\in v(S)$.

(c) $\Rightarrow$ (a): Clearly, $\sum(F^\times)^m\supseteq\sum S^m$. Conversely, let $\alpha\in\sum(F^\times)^m$. For every real valuation v we obtain from Lemma 3.2(b) and the assumption that $v(\alpha)\in mv(F^\times)=mv(S)=v(S^m)$. Now Theorem 6.2 (for the subgroup S^m) shows that $\alpha\in\sum S^m$. $\square$

7. Powers of sums vs. sums of powers

We now obtain our main result. It generalizes the remaining part of [Bec82, Th. 1.9] (which considers a single group $S=(F^\times)^n$, n even).

Theorem 7.1. *Suppose that F is a formally real field and let m be a positive integer. Let S_λ, $\lambda\in\Lambda$, be totally positive co-torsion subgroups of $F^\times$. Then*

$$\mathbb{E}^+\cdot\Big(\sum_{\lambda\in\Lambda}\sum S_\lambda\Big)^m=\sum_{\lambda\in\Lambda}\sum S_\lambda^m.$$

Proof. By Corollary 5.5,

$$\mathbb{E}^+\leqslant\sum_{\lambda\in\Lambda}\sum S_\lambda^m.$$

Next we show that $(\sum_{\lambda\in\Lambda}\sum S_\lambda)^m\leqslant\sum_{\lambda\in\Lambda}\sum S_\lambda^m$. To this end take $\alpha\in\sum_{\lambda\in\Lambda}\sum S_\lambda$. Let v be a real valuation on F. By Proposition 6.1 (with $\alpha_\lambda=\alpha$), $v(\alpha)\in v(S_\lambda)$ for some $\lambda\in\Lambda$. Hence $v(\alpha^m)\in mv(S_\lambda)=v(S_\lambda^m)$. Now the subgroups S_λ^m, $\lambda\in\Lambda$, of $F^\times$ are also totally positive and co-torsion. Therefore we may use Proposition 6.1 once again with $\alpha_\lambda=\alpha^m$ and with these subgroups to deduce that $\alpha^m\in\sum_{\lambda\in\Lambda}\sum S_\lambda^m$, as required. Consequently, in the assertion of the theorem, the left-hand side is contained in the right-hand side.

Conversely, to show that the right-hand side is contained in the left-hand side, take $\lambda(1),\dots,\lambda(r)\in\Lambda$ and $s_{i1},\dots,s_{it(i)}\in S_{\lambda(i)}$, $i=1,\dots,r$. By Corollary 3.3,

$$\frac{\sum_{i=1}^r\sum_{j=1}^{t(i)}s_{ij}^m}{(\sum_{i=1}^r\sum_{j=1}^{t(i)}s_{ij})^m}\in H_F^\times\cap\sum_{\lambda\in\Lambda}\sum S_\lambda\leqslant\mathbb{E}^+,$$

and the required inclusion follows. $\square$

Restricting to a single subgroup we obtain in particular:

Corollary 7.2. *Suppose that F is formally real and let m be a positive integer. Let S be a totally positive co-torsion subgroup of $F^\times$. Then*

$$\mathbb{E}^+ \cdot \Big(\sum S\Big)^m = \sum S^m.$$

The next proposition extends [Bec82, Prop. 3.1].

Proposition 7.3. *Suppose that F is formally real. Let S be a totally positive co-torsion subgroup of $F^\times$ and let m be a positive integer. The following conditions are equivalent*:

(a) $(\sum S)^m = \sum S^m$;
(b) $\mathbb{E}^+ = (\mathbb{E}^+)^m$.

Proof. (a) $\Rightarrow$ (b): By Corollary 7.2 and (a),

$$\mathbb{E}^+ \leqslant \Big(\sum S\Big)^m \leqslant \Big(\sum (F^\times)^2\Big)^m.$$

Furthermore, if $x \in F$ is totally positive and $x^m \in H_F^\times$, then by the definition of the real holomorphy ring, $x \in \mathbb{E}^+$. This proves that $\mathbb{E}^+ \leqslant (\mathbb{E}^+)^m$, whence (b).

(b) $\Rightarrow$ (a): By Corollary 5.5, $\mathbb{E}^+ \leqslant \sum S$. By (b), $\mathbb{E}^+ = (\mathbb{E}^+)^m \leqslant (\sum S)^m$. Now Corollary 7.2 gives (a). $\square$

Theorem 7.4. *Let S be a totally positive co-torsion subgroup of $F^\times$ and let m be a positive integer. The following conditions are equivalent*:

(a) *S^m is additively closed*;
(b) *S is additively closed and $\mathbb{E}^+ = (\mathbb{E}^+)^m$.*

Proof. First note that, by Theorem 2.2, both conditions imply that F is formally real. Hence there are no roots of unity $\neq 1$ in $\sum (F^\times)^2$. Therefore the map $x \mapsto x^m$ is injective on $\sum S$.

Now assume (a). By Corollary 7.2, $(\sum S)^m \leqslant \sum S^m = S^m$. It follows that $\sum S = S$. By Proposition 7.3, $\mathbb{E}^+ = (\mathbb{E}^+)^m$.

Conversely, assume (b). Then by Corollary 7.3, $S^m = (\sum S)^m = \sum S^m$. $\square$

As observed by Becker in [Bec82], the set of positive integers m satisfying $\mathbb{E}^+ = (\mathbb{E}^+)^m$ is the multiplicative semi-group generated by some set of prime numbers. Hence if (a) holds for some m, then it holds for any other positive integer m' whose prime divisors divide m. This extends results of Harman [Har82, Th. 5.7] and Jacob [Jac81, p. 261, Th.]. Theorem 7.4 also extends [Bec78, p. 63, Th. 26] (which deals with subgroups $S = (F^\times)^{2^k}$).

Finally, for a subgroup S of $F^\times$ we define the **S-Pythagoras number** $P_S(F)$ of F to be the minimal positive integer k—if such an integer exists—such that every element of $\sum S$ is the sum of at most k elements of S; when there is no such k we set $P_S(F) = \infty$. Now take $S = (F^\times)^2$ (so $P_S(F)$ is the usual Pythagoras number of F [Lam05, p. 395]). [Bec82, Th. 2.12] can then be rephrased as saying that for every $m \geqslant 1$, $P_S(F)$ is finite if and only if $P_{S^m}(F)$ is finite. Consequently, when $P_{(F^\times)^{2m}}(F)$ is finite for some m, it is finite for all m. We now extend the "if" part of the above equivalence to arbitrary totally positive co-torsion subgroups S.

Theorem 7.5. *Let F be a formally real field, let S be a totally positive co-torsion subgroup of $F^\times$, and let m be a positive integer. If $P_{S^m}(F) < \infty$, then $P_S(F) < \infty$. Moreover, in this case $P_S(F) \leqslant (P_{S^m}(F))^2$.*

Proof. Set $P = P_{S^m}(F)$. Given $s_1, \ldots, s_n \in S$, we choose $s'_1, \ldots, s'_P \in S$ such that $\sum_{i=1}^n s_i^m = \sum_{j=1}^P (s'_j)^m$. Also let

$$\epsilon = \frac{\sum_{i=1}^n s_i}{\sum_{j=1}^P s'_j}.$$

Corollary 3.3 implies that $\epsilon^m \in H_F^\times$. It therefore follows from the definition of H_F that $\epsilon \in H_F^\times$. Furthermore, $\epsilon \in \sum S \leqslant \sum (F^\times)^2$. By Corollary 7.2, $\epsilon \in \sum S^m$, so by assumption, ϵ is the sum of at most P elements of S^m ($\subseteq S$). Hence $\sum_{i=1}^n s_i$ is the sum of at most P^2 elements of S. $\square$

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